Error Exponents for Distributed Detection

Vien V. Mai, Youngmin Jeong, *Member, IEEE*, and Hyundong Shin, *Senior Member, IEEE*

***Abstract*—We consider distributed detection in wireless sensor networks with a multiple-antenna fusion center. Using the large deviation principle and random matrix theory, we analyze the detection performance of optimal hypothesis testing in terms of error exponents for the false alarm and miss detection probabilities.**

***Index Terms*—Error exponent, false alarm probability (FAP), hypothesis testing, large deviation, miss detection probability (MDP).**

I. Introduction

DISTRIBUTED detection in wireless sensor networks (WSNs) has been extensively studied for applications such as environmental monitoring, weather forecasts, health care, and home automation (see, e.g., [1]–[9] and references therein). In a traditional WSN, each sensor forwards its processed observation to a fusion center (FC) through parallel access channels (PACs) [1], [2], or a multiple access channel (MAC) [3]–[6]. The utilized bandwidth scales linearly with the number of sensors in the PACs, whereas this bandwidth is irrespective of the number of sensors in the MAC. However, the noisy received signal at the FC is generally not reliable for making a decision in the MAC due to the intrinsic properties of wireless channels (e.g., fading and interference). The multiple-antenna technology provides reliable communication without exceeding the costs in power and bandwidth. The recent advances in hardware technology also highly motivate to use large-scale multiple antennas at the FC as well as the dense deployment of low-cost sensors [4]–[9].

In this paper, we characterize the asymptotic detection performance. In particular, we show that under certain conditions, both the false alarm probability (FAP) and miss detection probability (MDP) decrease exponentially to zero (see Theorem 1). We then derive the error exponents for the FAP and MDP, which enable us to predict how difficult it will be to attain a certain level of detection reliability (see Theorem 2).

II. System Model and Methodology

A. System Model

We consider a distributed detection system where an $n_{\rm r}$-antenna FC collects data from $n_{\rm s}$ sensors during M-sample time. Let θ be a scalar-valued parameter (e.g., pressure, temperature, sound intensity, radiation level, pollution concentration, seismic activity, etc) to be detected. Then, for a binary hypothesis testing problem, the received signal at the FC under two hypotheses can be written as

$$\mathcal{H}_0 : \boldsymbol{Z} = \boldsymbol{H}\mathbf{A}\boldsymbol{W} + \boldsymbol{N} \tag{1}$$

$$\mathcal{H}_1 : \boldsymbol{Z} = \boldsymbol{H}\mathbf{A}\boldsymbol{G}\theta + \boldsymbol{H}\mathbf{A}\boldsymbol{W} + \boldsymbol{N} \tag{2}$$

where $\boldsymbol{G} = [\boldsymbol{g}_1 \ \dots \ \boldsymbol{g}_M] \in \mathbb{C}^{n_{\rm s}\times M}$ with $\boldsymbol{g}_m \sim \tilde{\mathcal{N}}_{n_{\rm s}}(\mathbf{0},\mathbf{I})$, $m = 1, 2, \dots, M$, is the Rayleigh-fading channel matrix between the source and sensors; $\boldsymbol{W} = [\boldsymbol{w}_1 \ \dots \ \boldsymbol{w}_M] \in \mathbb{C}^{n_{\rm s}\times M}$ with $\boldsymbol{w}_m \sim \tilde{\mathcal{N}}_{n_{\rm s}}\left(\mathbf{0},\sigma_{\rm w}^2\mathbf{I}\right)$ is the additive white Gaussian noise (AWGN) at the sensors; $\boldsymbol{N} = [\boldsymbol{n}_1 \ \dots \ \boldsymbol{n}_M] \in \mathbb{C}^{n_{\rm r}\times M}$ with $\boldsymbol{n}_m \sim \tilde{\mathcal{N}}_{n_{\rm r}}\left(\mathbf{0},\sigma_{\rm n}^2\mathbf{I}\right)$ is the AWGN matrix at the FC; $\boldsymbol{H} \in \mathbb{C}^{n_{\rm r}\times n_{\rm s}}$ is the Rayleigh-fading channel matrix between the sensors and the FC, which remains constant in the M-sample period; and $\mathbf{A} = \operatorname{diag}(\varrho_1, \dots, \varrho_{n_{\rm s}}) \in \mathbb{R}^{n_{\rm s}\times n_{\rm s}}$ is the diagonal matrix of amplification factors.[1] Note that all the random quantities $\boldsymbol{H}$, $\boldsymbol{G}$, $\boldsymbol{W}$, and $\boldsymbol{N}$ are statistically independent. The FC is assumed to have access to *perfect* knowledge of $\boldsymbol{H}$ but only *partial* statistical knowledge of $\boldsymbol{G}$ (i.e., mean and covariance of $\boldsymbol{G}$). The sensors do not have the channel knowledge and the network is subject to a total power constraint $P_{\rm tot}$ [8].[2] Hence, we set all the amplification factors $\varrho_1, \varrho_2, \dots, \varrho_{n_{\rm s}}$ to $\varrho/\sqrt{n_{\rm s}}$ where $\varrho = \sqrt{P_{\rm tot}}$.

B. Hypothesis Testing and Performance Measures

Let $\boldsymbol{z}_m$ be the mth column of $\boldsymbol{Z}$, where all the vectors $\boldsymbol{z}_m$ are mutual independent under both hypotheses. Then, $\boldsymbol{z}_m \sim \tilde{\mathcal{N}}_{n_{\rm r}}\left(\mathbf{0},\varrho^2\sigma_{\rm w}^2\mathbf{\Sigma}+\sigma_{\rm n}^2\mathbf{I}\right)$ under $\mathcal{H}_0$, and $\boldsymbol{z}_m \sim \tilde{\mathcal{N}}_{n_{\rm r}}\left(\mathbf{0},\varrho^2\left(\theta^2+\sigma_{\rm w}^2\right)\mathbf{\Sigma}+\sigma_{\rm n}^2\mathbf{I}\right)$ under $\mathcal{H}_1$ where $\mathbf{\Sigma} = \frac{1}{n_{\rm s}}\boldsymbol{H}\boldsymbol{H}^\dagger$ and $(\cdot)^\dagger$ denotes the transpose conjugate.

We consider the log-likelihood ratio (LLR) test:

$$\mathcal{T}_0 = \frac{1}{K}\log\frac{f(\boldsymbol{Z}|\boldsymbol{H},\mathcal{H}_1)}{f(\boldsymbol{Z}|\boldsymbol{H},\mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \mu \tag{3}$$

where $K = n_{\rm s}M$ and $f(\boldsymbol{Z}|\boldsymbol{H},\mathcal{H}_i)$ is the probability density function of $\boldsymbol{Z}$ given the channel matrix $\boldsymbol{H}$ under $\mathcal{H}_i$.[3] By applying the matrix inversion lemma to (3), we arrive at the equivalent optimal test:

$$\mathcal{T}_1 = \frac{1}{K}\operatorname{tr}\left(\boldsymbol{Z}^\dagger\left[\mathbf{\Phi}\left(\varrho^2\theta^2\mathbf{\Sigma}\right)^{-1}\mathbf{\Phi}+\mathbf{\Phi}\right]^{-1}\boldsymbol{Z}\right) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \xi \tag{4}$$

Manuscript received August 18, 2015; revised October 14, 2015; and accepted November 8, 2015. Date of publication November 23, 2015; date of current version January 7, 2016. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (MSIP) (No. 2013-R1A1A2-019963 and No. 2009-0083495). The associate editor coordinating the review of this paper and approving it for publication was Wei Xu.

The authors are with the Department of Electronics and Radio Engineering, Kyung Hee University, Yongin 17104, Korea (e-mail: hshin@khu.ac.kr).

Digital Object Identifier 10.1109/LCOMM.2015.2502943

[1]The notations $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers; $\mathbf{I}$ denotes the identity matrix; and $\tilde{\mathcal{N}}_n(\mathbf{m},\mathbf{\Sigma})$ denotes the $n\times 1$ complex Gaussian vector with mean vector $\mathbf{m}\in\mathbb{C}^n$ and covariance matrix $\mathbf{\Sigma}\in\mathbb{C}^{n\times n}$.

[2]The power allocation problem in the dense WSNs with multiple antennas at FC is a still open problem. However, it is difficult to track the all instantaneous channel state information at the sensors under the dense deployment of low-cost sensors as well as a large number of antennas at the FC in practise.

[3]The threshold μ is chosen to guarantee a fixed false alarm rate under the Neyman-Pearson approach or equivalently equals to $\frac{1}{K}\log(\pi_0/\pi_1)$ where π_i is *a prior* probability of the hypothesis $\mathcal{H}_i$, which is minimizing the overall error probability under the Bayesian approach [5].

where $\mathrm{tr}(\cdot)$ is the trace operator; $\mathbf{\Phi} = \varrho^2\sigma_{\mathrm{w}}^2\mathbf{\Sigma} + \sigma_{\mathrm{n}}^2\mathbf{I}$; and

$$\xi = \mu + \frac{1}{n_{\mathrm{s}}}\log\frac{\det\left(\mathbf{\Phi} + \varrho^2\theta^2\mathbf{\Sigma}\right)}{\det\left(\mathbf{\Phi}\right)}. \tag{5}$$

Taking the eigenvalue decomposition of $\mathbf{\Sigma}$ as $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$ where $\mathbf{U}$ is a unitary matrix whose columns contain eigenvectors of $\mathbf{\Sigma}$, and $\mathbf{\Lambda} = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_{n_{\mathrm{r}}}\right)$ is a diagonal matrix containing eigenvalues of $\mathbf{\Sigma}$. After some manipulations, the optimal test $\mathcal{T}_1$ can be expressed as

$$\mathcal{T}_1 = \frac{1}{K}\sum_{i=1}^{n_{\mathrm{r}}}\frac{\varrho^2\theta^2\lambda_i\left\|\boldsymbol{x}_i\right\|^2}{\left(\varrho^2\sigma_{\mathrm{w}}^2\lambda_i + \sigma_{\mathrm{n}}^2\right)\left(\varrho^2\left(\sigma_{\mathrm{w}}^2 + \theta^2\right)\lambda_i + \sigma_{\mathrm{n}}^2\right)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \xi \tag{6}$$

where $\boldsymbol{x}_i$ is the ith row of $\boldsymbol{X} \triangleq \mathbf{U}^\dagger\boldsymbol{Z}$. Note that, due to the unitary invariance of Gaussian distribution, the mth column of $\boldsymbol{X}$ is a zero mean complex Gaussian vector with the covariance matrix $\varrho^2\sigma_{\mathrm{w}}^2\mathbf{\Lambda}+\sigma_{\mathrm{n}}^2\mathbf{I}$ under $\mathcal{H}_0$ and $\varrho^2\left(\sigma_{\mathrm{w}}^2 + \theta^2\right)\mathbf{\Lambda}+\sigma_{\mathrm{n}}^2\mathbf{I}$ under $\mathcal{H}_1$. Thus, it follows that $\left\|\boldsymbol{x}_i\right\|^2$ is the sum of M squared of independent and identical distributed (i.i.d.) complex normal random variables. To simplify the subsequent calculations, we denote $\gamma_0 \triangleq \theta^2/\sigma_{\mathrm{w}}^2$, $\gamma \triangleq P_{\mathrm{tot}}\sigma_{\mathrm{w}}^2/\sigma_{\mathrm{n}}^2$, and $\gamma_1 \triangleq \gamma\left(1 + \gamma_0\right)$.

We define $\alpha^{(K)} = \mathbb{P}\left\{\mathcal{T}_1 > \xi|\mathcal{H}_0\right\}$ and $\beta^{(K)} = \mathbb{P}\left\{\mathcal{T}_1 \leq \xi|\mathcal{H}_1\right\}$ as the FAP and MDP, respectively. For the Bayesian criterion, we also define the detection error probability (DEP) in terms of $\alpha^{(K)}$ and $\beta^{(K)}$ as follows:

$$P_{\mathrm{e}} = \pi_0\alpha^{(K)} + \pi_1\beta^{(K)}. \tag{7}$$

In this paper, we are interested in the *error exponents* for the FAP and MDP which are defined respectively as

$$\mathcal{K}_\alpha = \lim_{K\to\infty} -\frac{1}{K}\log\alpha^{(K)} \tag{8}$$

$$\mathcal{K}_\beta = \lim_{K\to\infty} -\frac{1}{K}\log\beta^{(K)}. \tag{9}$$

Hence, the error exponent for the DEP of the Bayesian criterion is given by [10]

$$\mathcal{K}_{\mathrm{e}} = \lim_{K\to\infty} -\frac{1}{K}\log P_{\mathrm{e}} = \min\left(\mathcal{K}_\alpha, \mathcal{K}_\beta\right). \tag{10}$$

Theorem 1 (FAP and MDP Decay Trends): Consider a detection system with $n_{\mathrm{r}} \to \infty$ in such a way that $n_{\mathrm{s}}/n_{\mathrm{r}} \to \tau$ and $M/n_{\mathrm{r}} \to \kappa$. Let $\lambda^+ = \left(1 + 1/\sqrt{\tau}\right)^2$. Then, for a given threshold ξ such that

$$\frac{\gamma\gamma_0\lambda^+}{\tau\left(\gamma_1\lambda^+ + 1\right)} < \xi < \frac{\gamma\gamma_0\lambda^+}{\tau\left(\gamma\lambda^+ + 1\right)}, \tag{11}$$

the FAP and MDP decrease exponentially to zero.

Proof: We begin by bounding the FAP as follows:

$$\begin{aligned}
&\mathbb{P}\left\{\mathcal{T}_1 > \xi|\mathcal{H}_0\right\} \\
&= \mathbb{P}\left\{\frac{1}{K}\sum_{i=1}^{n_{\mathrm{r}}}\frac{\gamma\gamma_0\lambda_i}{\gamma_1\lambda_i + 1}\frac{\zeta_i}{2} > \xi\right\} \\
&\leq 1 - \mathbb{P}\left\{\max_{1\leq i\leq n_{\mathrm{r}}}\frac{\gamma\gamma_0\lambda_i}{\gamma_1\lambda_i + 1}\frac{\zeta_i}{2} \leq \xi\tau M\right\} \\
&= 1 - \exp\left(\sum_{i=1}^{n_{\mathrm{r}}}\log\left(1 - \mathbb{P}\left\{\zeta_i \geq 2\xi\epsilon_i\tau M\right\}\right)\right)
\end{aligned} \tag{12}$$

where $\epsilon_i = \frac{\gamma_1\lambda_i+1}{\gamma\gamma_0\lambda_i}$, and ζ_i denotes the chi-squared random variable with $2M$ degrees of freedom for $i = 1, \ldots, n_{\mathrm{r}}$. We observe that for all $t > 0$

$$\begin{aligned}
\mathbb{P}\left\{\zeta_i \geq 2\xi\epsilon_i\tau M\right\} &\overset{(a)}{\leq} e^{-2\xi\epsilon_i\tau M t}\,\mathbb{E}\left\{e^{t\zeta_i}\right\} \\
&\overset{(b)}{=} e^{-M(2\epsilon_i\xi\tau t + \log(1-2t))}
\end{aligned} \tag{13}$$

where (a) follows from the Markov's inequality; and (b) follows from the moment generating function of the Chi-squared random variables. We note that the function $f_{\epsilon_i}(t) \triangleq 2\epsilon_i\xi\tau t + \log\left(1 - 2t\right)$ is concave for all $t > 0$, and it is maximized at $t^\star = \frac{1}{2}\left(1 - \frac{1}{\tau\epsilon_i\xi}\right)$. For a tight bound, we choose $t = t^\star$ such that $\inf_i \frac{1}{2}\left(1 - \frac{1}{\tau\epsilon_i\xi}\right) > 0$, or equivalently, $\xi > \frac{\gamma\gamma_0\lambda^+}{\tau(\gamma_1\lambda^+ + 1)}$. Substituting t into (13) and combining with (12), for any given $\delta > 0$, if n_{r} is sufficiently large such that $n_{\mathrm{s}}/n_{\mathrm{r}} \to \tau$ and $M/n_{\mathrm{r}} \to \kappa$, we get

$$\begin{aligned}
\mathbb{P}\left\{\mathcal{T}_1 > \xi|\mathcal{H}_0\right\} &\leq 1 - \exp\left(\sum_{i=1}^{n_{\mathrm{r}}}\log\left(1 - e^{-Mf_{\epsilon_i}(t^\star)}\right)\right) \\
&\overset{(a)}{\leq} 1 - \exp\left(\kappa^{-1}M\log\left(1 - e^{-Mf_{\epsilon_{i^\star}}(t^\star)}\right)\right) \\
&\overset{(b)}{=} 1 - \exp\left(-\kappa^{-1}Me^{-Mf_{\epsilon_{i^\star}}(t^\star)}\left(1 + o\left(1\right)\right)\right) \\
&\overset{(c)}{\leq} \delta, \quad \forall\delta > 0
\end{aligned} \tag{14}$$

where $o\left(\cdot\right)$ is the Bachmann-Landau notation; (a) follows from $i^\star = \arg\min_i f_{\epsilon_i}\left(t^\star\right)$; (b) follows from the Taylor's expansion of $\log\left(1 - x\right)$; and (c) follows from the fact that $x > \log\left(x\right) + 1$ for all $x \in (1, \infty)$. From which we complete the proof for the FAP by letting $\delta \to 0$. We omitted the proof for the MDP, which can be found using the similar steps of the proof for the FAP. □

III. Error Exponents

In this section, we analyze the error exponents for the FAP and MDP.

Theorem 2 (Error Exponents): Consider a detection system with $n_{\mathrm{r}} \to \infty$ in such a way that $n_{\mathrm{s}}/n_{\mathrm{r}} \to \tau$. Then, for a given threshold ξ such that

$$\frac{\gamma_0}{4\gamma\left(1 + \gamma_0\right)^2}\mathcal{F}\left(\frac{\gamma_1}{\tau}, \tau\right) < \xi < \frac{\gamma_0}{4\gamma}\mathcal{F}\left(\frac{\gamma}{\tau}, \tau\right), \tag{15}$$

the error exponents for the FAP $\mathcal{K}_\alpha$ and MDP $\mathcal{K}_\beta$ defined in (8) and (9) are given by

$$\mathcal{K}_\alpha = \mathcal{K}\left(\gamma_2, \gamma_1, \xi s_0^\star; \tau\right) \tag{16}$$

$$\mathcal{K}_\beta = \mathcal{K}\left(\gamma_3, \gamma, \xi s_1^\star; \tau\right) \tag{17}$$

where $\mathcal{K}\left(x, z, s; \tau\right)$ is given in (18); $\gamma_2 = \gamma\left(1 + \left(1 - s_0^\star\right)\gamma_0\right)$; $\gamma_3 = \gamma\left(1 - s_1^\star\gamma_0\right)$; and $s_0^\star$ and $s_1^\star$ are

$$s_0^\star = \begin{cases} \frac{\gamma_1\lambda^+ + 1}{\gamma\gamma_0\lambda^+}, & \xi > \frac{\gamma\gamma_0\lambda^+}{\sqrt{\tau}} \\ 1 + \frac{1}{\gamma_0}, & \xi = \frac{\gamma\gamma_0}{\tau} \\ 1 + \frac{\gamma - \bar{\gamma}}{\gamma\gamma_0}, & \text{otherwise} \end{cases} \tag{19}$$

$$s_1^\star = \begin{cases} \frac{\gamma\lambda^+ + 1}{\gamma\gamma_0\lambda^+}, & \xi > \frac{\gamma\gamma_0\lambda^+}{\sqrt{\tau}} \\ \frac{1}{\gamma_0}, & \xi = \frac{\gamma\gamma_0}{\tau} \\ \frac{\gamma - \bar{\gamma}}{\gamma\gamma_0}, & \text{otherwise} \end{cases} \tag{20}$$

$$\mathcal{K}(x,z,s;\tau)=s+\log\frac{\tau+x-\frac{\tau}{4}\mathcal{F}\left(\frac{x}{\tau},\tau\right)}{\tau+z-\frac{\tau}{4}\mathcal{F}\left(\frac{z}{\tau},\tau\right)}+\frac{1}{\tau}\log\frac{1+x-\frac{1}{4}\mathcal{F}\left(\frac{x}{\tau},\tau\right)}{1+z-\frac{1}{4}\mathcal{F}\left(\frac{z}{\tau},\tau\right)}-\left(\frac{\mathcal{F}\left(\frac{x}{\tau},\tau\right)}{4x}-\frac{\mathcal{F}\left(\frac{z}{\tau},\tau\right)}{4z}\right)\log e \tag{18}$$

where $\bar{\gamma}$ is the unique solution to the fixed-point equation

$$\xi-\frac{\gamma\gamma_0}{4\bar{\gamma}^2}\mathcal{F}\left(\frac{\bar{\gamma}}{\tau},\tau\right)=0 \tag{21}$$

with

$$\mathcal{F}(x,z)\triangleq\left(\sqrt{x\left(1+\sqrt{z}\right)^2+1}-\sqrt{x\left(1-\sqrt{z}\right)^2+1}\right)^2. \tag{22}$$

Proof: We begin by defining the logarithmic moment generating function (LMGF) $\Lambda_i^{(n_{\mathrm{r}})}(s)$ and its limiting LMGF $\Lambda_i(s)$ for $i=0,1$ as

$$\Lambda_i^{(n_{\mathrm{r}})}(s)\triangleq\log\mathbb{E}\left\{e^{s\mathcal{T}_1}|\mathcal{H}_i\right\} \tag{23}$$

$$\Lambda_i(s)\triangleq\lim_{n_{\mathrm{r}}\to\infty}\frac{1}{K}\Lambda_i^{(n_{\mathrm{r}})}(Ks). \tag{24}$$

Under the hypothesis $\mathcal{H}_0$, we have

$$\begin{aligned}\Lambda_0(s)&=\lim_{n_{\mathrm{r}}\to\infty}\frac{1}{K}\log\left(\mathbb{E}\left\{\exp\left(\sum_{i=1}^{n_{\mathrm{r}}}\frac{s\gamma\gamma_0\lambda_i}{\gamma_1\lambda_i+1}\frac{\zeta_i}{2}\right)\right\}\right)\\&=\lim_{n_{\mathrm{r}}\to\infty}-\frac{1}{n_{\mathrm{s}}}\sum_{i=1}^{n_{\mathrm{r}}}\log\left(1-\frac{s\gamma\gamma_0\lambda_i}{\gamma_1\lambda_i+1}\right)\\&\xrightarrow{\text{a.s}}-\frac{1}{\tau}\int_{\lambda^-}^{\lambda^+}\log\left(1-\frac{s\gamma\gamma_0\lambda}{\gamma_1\lambda+1}\right)f^\star(\lambda;\tau)\,d\lambda\end{aligned} \tag{25}$$

for all $s\leq\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}$, otherwise $\Lambda_0(s)\to+\infty$.[4] Thus, $\Lambda_0(s)$ is an extended real number and lower-semicontinuous function. Because of the bounded derivative at the boundary point, $\Lambda_0(s)$ dose not satisfy the steepness condition, and hence the Gärtner-Ellis theorem [10, Theorem 2.3.6] is not directly applicable. However, it has been shown that the hypothesis testing $\mathcal{T}_1$ satisfies the large deviation (LD) principle with a good rate function $\Lambda_i^\star(\cdot)$ which can be characterized in terms of the Fenchel-Legendre transform of $\Lambda_i(s)$ [12]:

$$\Lambda_i^\star(x)=\sup_{s\in\mathbb{R}}\left\{xs-\Lambda_i(s)\right\},\quad x\in\mathbb{R}. \tag{26}$$

Let $\mathcal{G}\triangleq\{x|x>\xi\}$, then for a given threshold ξ such that $\xi>\lim_{n_{\mathrm{r}}\to\infty}\mathbb{E}\{\mathcal{T}_1|\mathcal{H}_0\}$, the error exponent for FAP is given by[5]

$$\mathcal{K}_\alpha=\inf_{x\in\mathcal{G}}\Lambda_0^\star(x)=\Lambda_0^\star(\xi). \tag{27}$$

Let $s_0^\star$ be the optimal solution s in (26) under $\mathcal{H}_0$ and $\dot{\omega}(s)$ be the first order derivative of $\omega(s)\triangleq\xi s-\Lambda_0(s)$ with respect to s. Since $\Lambda_0(s)$ is strictly convex on $s\in\left(-\infty,\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}\right)$, there are obviously two possible solutions for $s_0^\star$:

1) If $\dot{\omega}(s)>0$ on $s\in\left(-\infty,\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}\right)$, then $s_0^\star=\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}$.
2) Otherwise, $s_0^\star\in\left(-\infty,\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}\right)$ is the solution which satisfies $\dot{\omega}(s)=0$.

For $\bar{\gamma}=\gamma(1+(1-s)\gamma_0)\neq0$, we have

$$\begin{aligned}\dot{\omega}(s)&=\xi-\frac{\gamma\gamma_0}{\bar{\gamma}}\left(1-\int_{\lambda^-}^{\lambda^+}\frac{1}{\bar{\gamma}\lambda+1}f^\star(\lambda;\tau)\,d\lambda\right)\\&=\xi-\frac{\gamma\gamma_0}{4\bar{\gamma}^2}\mathcal{F}\left(\frac{\bar{\gamma}}{\tau},\tau\right)\end{aligned} \tag{28}$$

where the last equality follows from [11, eq. (2.52)]. For $\bar{\gamma}=0$, i.e., $s=1+\gamma_0^{-1}$, $\dot{\omega}(s)$ boils down to

$$\dot{\omega}\left(1+\gamma_0^{-1}\right)=\xi-\tau^{-1}\gamma\gamma_0. \tag{29}$$

When $s_0^\star\neq1+\gamma_0^{-1}$, evaluating $\dot{\omega}(s)$ at the boundary point $s=\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}$, we get

$$\dot{\omega}\left(\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}\right)=\xi-\frac{\gamma\gamma_0\lambda^+}{\sqrt{\tau}} \tag{30}$$

from which we arrive at the first case in (19). Since $1+\gamma_0^{-1}<\frac{\gamma_1\lambda^++1}{\gamma\gamma_0\lambda^+}$, we can exclude the first possibility immediately, and hence $s_0^\star=1+\gamma_0^{-1}$ if and only if $\xi=\tau^{-1}\gamma\gamma_0$. Beyond these two extreme cases, the optimal $s_0^\star$ occurs at $\dot{\omega}(s)=0$.

We now derive $\Lambda_0^\star(\xi)$ defined in (26) by noting that

$$\begin{aligned}\Lambda_0(s_0^\star)&=\tau^{-1}\int_{\lambda^-}^{\lambda^+}\log(1+\gamma_1\lambda)f^\star(\lambda;\tau)\,d\lambda\\&\quad-\tau^{-1}\int_{\lambda^-}^{\lambda^+}\log(1+\gamma_2\lambda)f^\star(\lambda;\tau)\,d\lambda\\&=\tau^{-1}\mathcal{V}\left(\frac{\gamma_1}{\tau},\tau\right)-\tau^{-1}\mathcal{V}\left(\frac{\gamma_2}{\tau},\tau\right)\end{aligned} \tag{31}$$

where $\mathcal{V}(\cdot,\cdot)$ is given in [11, eq. (3.140)] as

$$\begin{aligned}\mathcal{V}(x,z)&\triangleq z\log\left(1+x-\frac{1}{4}\mathcal{F}(x,z)\right)\\&\quad+\log\left(1+xz-\frac{1}{4}\mathcal{F}(x,z)\right)-\frac{\log e}{4x}\mathcal{F}(x,z).\end{aligned} \tag{32}$$

From (26), (27), (31), and (32), we arrive at the desired result in (16) with the threshold ξ chosen from

$$\lim_{n_{\mathrm{r}}\to\infty}\mathbb{E}\{\mathcal{T}_1|\mathcal{H}_0\}=\lim_{n_{\mathrm{r}}\to\infty}\frac{1}{n_{\mathrm{s}}}\sum_{i=1}^{n_{\mathrm{r}}}\frac{\gamma\gamma_0\lambda_i}{\gamma_1\lambda_i+1}\xrightarrow{\text{a.s}}\frac{\gamma_0\,\mathcal{F}\left(\frac{\gamma_1}{\tau},\tau\right)}{4\gamma\left(1+\gamma_0\right)^2} \tag{33}$$

$$\lim_{n_{\mathrm{r}}\to\infty}\mathbb{E}\{\mathcal{T}_1|\mathcal{H}_1\}=\lim_{n_{\mathrm{r}}\to\infty}\frac{1}{n_{\mathrm{s}}}\sum_{i=1}^{n_{\mathrm{r}}}\frac{\gamma\gamma_0\lambda_i}{\gamma\lambda_i+1}\xrightarrow{\text{a.s}}\frac{\gamma_0\,\mathcal{F}\left(\frac{\gamma}{\tau},\tau\right)}{4\gamma}. \tag{34}$$

[4]The notation $\xrightarrow{\text{a.s}}$ denotes the almost sure convergence. The function $f^\star(\lambda;\tau)$ denotes the Marčenko-Pastur law with parameter $1/\tau$, which is the asymptotic empirical eigenvalue distribution of $\mathbf{\Sigma}$, with $\lambda^-=\left(1-1/\sqrt{\tau}\right)^2$ and $\lambda^+=\left(1+1/\sqrt{\tau}\right)^2$, respectively [11].

[5]Note that in hypothesis testing, the set $\mathcal{G}\subset\mathbb{R}$ mostly satisfies the so-called I-continuity, i.e., $\inf_{x\in\mathcal{G}^\circ}\Lambda_0^\star(x)=\inf_{x\in\underline{\mathcal{G}}}\Lambda_0^\star(x)$ with $\mathcal{G}^\circ$ and $\underline{\mathcal{G}}$ are the interior and closure of $\mathcal{G}$, respectively [6], [10]. Let $T_0\triangleq\lim_{n_{\mathrm{r}}\to\infty}\mathbb{E}\{\mathcal{T}_1|\mathcal{H}_0\}$ and $T_1\triangleq\lim_{n_{\mathrm{r}}\to\infty}\mathbb{E}\{\mathcal{T}_1|\mathcal{H}_1\}$. Then it can be shown that $\mathcal{K}_\alpha=0$ for $\xi\leq T_0$ and $\mathcal{K}_\beta=0$ for $\xi\geq T_1$, and $\Lambda_0^\star(x)$ is a nondecreasing function for $x\in(T_0,\infty)$ while $\Lambda_1^\star(x)$ is a nonincreasing function for $x\in(-\infty,T_1)$. Thus, the infimum over the set of interest is attained at the boundary point $x=\xi$.

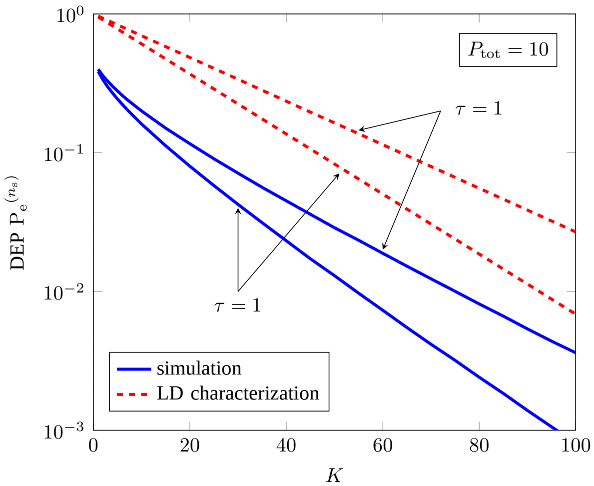

Fig. 1. DEP P_e and the theoretical LD characterization $e^{-K\mathcal{K}_e}$ for the Bayesian testing as a function of K when $P_{\rm tot} = 10$, $\theta = 1$, $\sigma_n = 1$, $\sigma_w = 1$, $\pi_0 = \pi_1 = 0.5$, and $\tau = 0.5, 1$.

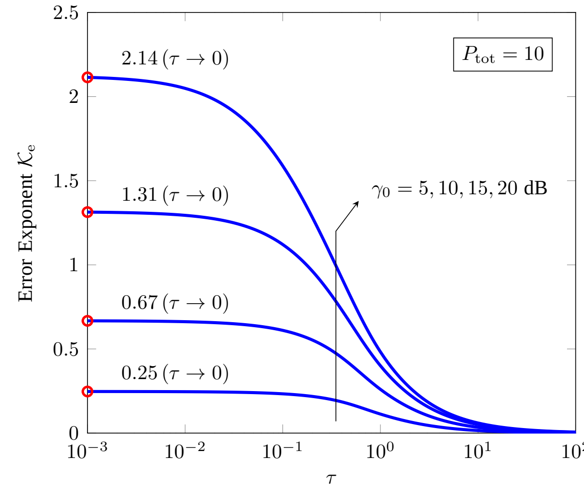

Fig. 2. Error exponent for the DEP $\mathcal{K}_e$ as a function of τ when $P_{\rm tot} = 10$, $\theta = 1$, $\sigma_n = 1$, $\pi_0 = \pi_1 = 0.5$, and $\gamma_0 = 5, 10, 15, 20$ dB.

We omitted the proof for the MDP $\mathcal{K}_\beta$, which can be found using the similar steps of the proof for the FAP $\mathcal{K}_\alpha$. □

Remark 1: As can be seen from Theorem 2, the limiting value κ does not affect error exponents.

Corollary 1 (Limiting Error Exponents): As $\tau \to 0$, the error exponents $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ in Theorem 2 approach to

$$\mathcal{K}_\alpha^\downarrow = \left(1 + \gamma_0^{-1}\right)\xi + \log\left(\gamma_0 \xi^{-1}\right) - \log\left(1 + \gamma_0\right) - 1 \quad (35)$$

$$\mathcal{K}_\beta^\downarrow = \gamma_0^{-1}\xi + \log\left(\gamma_0 \xi^{-1}\right) - 1. \quad (36)$$

Proof: These results can be verified from the fact that

$$\lim_{\tau \to 0} \frac{\mathcal{F}\left(\frac{x}{\tau}, \tau\right)}{4x} = 1 \quad (37)$$

$$\lim_{\tau \to 0} \frac{1}{\tau} \log\left(1 + x - \frac{1}{4}\mathcal{F}\left(\frac{x}{\tau}, \tau\right)\right) = 1. \quad (38)$$

Using (37), we can explicitly find that $s_0^\star = 1 + \gamma_0^{-1} - \xi^{-1}$ and $s_1^\star = \gamma_0^{-1} - \xi^{-1}$. Note that as $\tau \to 0$, $s_0^\star$ and $s_1^\star$ include the second cases in (19) and (20), respectively, while the first cases in (19) and (20) are excluded due to $\xi \to \infty$. We also note that, as $\tau \to \infty$, the error exponents $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ in Theorem 2 approach to zero since $\lim_{\tau\to\infty} \mathcal{F}\left(\frac{x}{\tau}, \tau\right) = 0$. □

Remark 2: The amplification factors are *static* with respect to the time samples. Note that the equally fixed ϱ_i is not necessary for all results obtained in this paper and hence, it suffices to have all ϱ_i scales as $1/\sqrt{n_s}$.

IV. Numerical Results and Discussion

Fig. 1 shows the DEP $\mathrm{P_e}$ and the theoretical LD characterization $e^{-K\mathcal{K}_e}$ [6] for the Bayesian testing as a function of K when $P_{\rm tot} = 10$, $\theta = 1$, $\sigma_n = 1$, $\sigma_w = 1$, $\pi_0 = \pi_1 = 0.5$, and $\tau = 0.5, 1$. It can be seen that the DEP decays exponentially with K, and the slopes of the DEP curves agree with the theoretical results as K grows large enough. In this example, the error exponents for the DEP are equal to $\mathcal{K}_e = 0.05$ and 0.0362 for $\tau = 0.5$ and 1, respectively.

The properties of the error exponents can be ascertained by referring to Fig. 2 where the error exponent for the DEP $\mathcal{K}_e$ is depicted as a function of τ when $P_{\rm tot} = 10$, $\theta = 1$, $\sigma_n = 1$, $\pi_0 = \pi_1 = 0.5$, and $\gamma_0 = 5, 10, 15, 20$ dB. We can see that the error exponent $\mathcal{K}_e$ monotonically decreases with τ and quickly reaches the limiting values as in Corollary 1. It can be attributed from the fact that for a given number of sensors, small τ gives higher number of degrees of freedom for reliable detection. As expected, the error exponents $\mathcal{K}_e$ approach to 0 as $\tau \to \infty$ while the limiting error exponents defined as $\mathcal{K}_e^\downarrow \triangleq \min\left(\mathcal{K}_\alpha^\downarrow, \mathcal{K}_\beta^\downarrow\right)$ are equal to $\mathcal{K}_e^\downarrow = 0.25$, 0.67, 1.31 and 2.14 for $\gamma_0 = 5$, 10, 15 and 20 dB, respectively.

References

[1] R. Viswanathan and P. K. Varshney, "Distributed detection with multiple sensors: Part I–fundamentals," *Proc. IEEE*, vol. 85, no. 1, pp. 54–63, Jan. 1997.

[2] J.-F. Chamberland and V. V. Veeravalli, "How dense should a sensor network be for detection with correlated observations?" *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 5099–5106, Nov. 2006.

[3] D. Ciuonzo, G. Romano, and P. Salvo Rossi, "Optimality of received energy in decision fusion over Rayleigh fading diversity MAC with non-identical sensors," *IEEE Trans. Signal Process.*, vol. 61, no. 1, pp. 22–27, 2013.

[4] K. Liu and A. M. Sayeed, "Type-based decentralized detection in wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 55, no. 5, pp. 1899–1910, May 2007.

[5] W. Li and H. Dai, "Distributed detection in wireless sensor networks using a multiple access channel," *IEEE Trans. Signal Process.*, vol. 55, no. 3, pp. 822–833, Mar. 2007.

[6] K. Cohen and A. Leshem, "Performance analysis of likelihood-based multiple access for detection over fading channels," *IEEE Trans. Inf. Theory*, vol. 59, no. 4, pp. 2471–2481, Apr. 2013.

[7] D. Ciuonzo, P. Salvo Rossi, and S. Dey, "Massive MIMO channel-aware decision fusion," *IEEE Trans. Signal Process.*, vol. 63, no. 3, pp. 604–619, Feb. 2015.

[8] F. Jiang, J. Chen, A. L. Swindlehurst, and J. A. López-Salcedo, "Massive MIMO for wireless sensing with a coherent multiple access channel," *IEEE Trans. Signal Process.*, vol. 63, no. 12, pp. 3005–3017, Jun. 2015.

[9] I. Nevat, G. W. Peters, and I. B. Collings, "Distributed detection in sensor networks over fading channels with multiple antennas at the fusion centre," *IEEE Trans. Signal Process.*, vol. 62, no. 3, pp. 671–683, Feb. 2014.

[10] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*. New York: Springer, 1998, vol. 2.

[11] A. M. Tulino and S. Verdú, *Random matrix theory and wireless communications*. Delft, The Netherlands: Now Publishers Inc, 2004.

[12] F. G. B. Bercu and A. Rouault, "Large deviations for quadratic forms of stationary Gaussian processes," *Stoch. Process. Appl.*, vol. 71, pp. 75–79, 1997.